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THE MAXIMAL AND MINIMAL 2-CORRELATION OF A CLASS OF 1-DEPENDENT 0–1 VALUED PROCESSES

BY

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ABSTRACT

We compute the maximal and minimal value of $P[X_N = X_{N+1} = 1]$ for fixed $P[X_N = 1]$, where $(X_N)_{N \in \mathbf{Z}}$ is a 0–1 valued 1-dependent process obtained by a coding of an i.i.d.-sequence of uniformly [0,1] distributed random variables with a subset of the unit square.

1. Introduction

A stationary, 0–1 valued, stochastic process $(X_N)_{N \in \mathbf{Z}}$ is *1-dependent* if

$$P[X_{-N} = i_{-N}, \ldots, X_{-1} = i_{-1}, X_1 = i_1, \ldots, X_N = i_N]$$

$$= P[X_{-N} = i_{-N}, \ldots, X_{-1} = i_{-1}] \cdot P[X_1 = i_1, \ldots, X_N = i_N]$$

for all $N \geqq 1$ and for all $i_{-N}, \ldots, i_{-1}, i_1, \ldots, i_N \in \{0, 1\}$.

For quite a long time it seemed to be folklore to conjecture that each 1-dependent process is an indicator process (we will define that), but recently Aaronson and Gilat ([AG]) found a counterexample of a 1-dependent process that is not an indicator process. A paper by Aaronson, Gilat, Keane and De Valk [AGKV] on a two-parameter family of such counterexamples has been written.

Let J be the unit interval, J^2 the unit square, let λ and μ be Lebesgue measure on J and J^2 resp. and let A be the collection of μ-measurable sets in J^2.

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Let $(U_N)_{N\in\mathbf{Z}}$ be an i.i.d. sequence of random variables uniformly distributed over J. Define for each $A \in \mathcal{A}$ the corresponding *indicator process* $(X_N)_{N\in\mathbf{Z}}$:

$$X_N := \begin{cases} 0, & \text{if } (U_N, U_{N+1}) \notin A, \\ 1, & \text{if } (U_N, U_{N+1}) \in A. \end{cases}$$

It is easy to see that each indicator process is a 1-dependent process and that

$$P[X_N = 1] = \mu(A).$$

From now on we reserve α for the Lebesgue measure of A (thus $\alpha = \mu(A)$ is the probability of a one).

In 1971 Katz [Ka] computed (translated to our terminology) the maximal value of a 2-correlation $P[X_N = X_{N+1} = 1]$ over the class of indicator processes for fixed α.

Finke [F] (1982) was the first to interpret Katz's mathematical objects as correlations in stochastic processes.

Recently Gandolfi, Keane and De Valk [GKV] proved a more general result about the maximal value of a 2-correlation over the class of 1-dependent processes. They computed that the 2-correlation (for fixed probability of a one) has the same upper bound over the class of 1-dependent processes as over the class of indicator processes.

Further, they proved that there exists a unique 1-dependent process with this 2-correlation if the probability of a one is not $\frac{1}{2}$. If the probability of a one is $\frac{1}{2}$, there exist exactly two 1-dependent processes with this 2-correlation (and both are indicator processes). So, the conjecture mentioned in the beginning of this section does not hold in general, but is true for these extremal cases.

In this paper we will compute the minimal 2-correlation for all indicator processes. For $\alpha \notin (\frac{1}{3}, \frac{2}{3})$ we have been able to compute the minimal 2-correlation for 1-dependent processes, finding the same lower bound ([GKV]).

For $\alpha \notin (\frac{1}{4}, \frac{3}{4})$ we know that there exists a unique process with this 2-correlation ([GKV]).

2. Basic properties

For $A \in \mathcal{A}$ we define the *horizontal and vertical sections* H_A and V_A:

$$H_A(y) := \lambda\{x \in J : (x, y) \in A\}, \qquad y \in J,$$

$$V_A(x) := \lambda\{y \in J : (x, y) \in A\}, \qquad x \in J,$$

and we define I_A:

$$I_A := \int_0^1 H_A(x) V_A(x) d\lambda(x).$$

LEMMA 1. *The 2-correlation $P[X_N = X_{N+1} = 1]$ of an indicator process is equal to I_A.*

PROOF. Directly from the definitions,

$$\begin{aligned} P[X_N = X_{N+1} = 1] &= P[(U_N, U_{N+1}) \in A, (U_{N+1}, U_{N+2}) \in A] \\ &= \int_0^1 P[(U_N, U_{N+1}) \in A, (U_{N+1}, U_{N+2}) \in A \mid U_{N+1} = x] d\lambda(x) \\ &= \int_0^1 H_A(x) V_A(x) d\lambda(x) \\ &= I_A. \end{aligned}$$ □

We define the *maximal and minimal 2-correlations* of an indicator process by

$$\mathrm{Max}(\alpha) := \sup\{I_A : A \in \mathcal{A}, \mu(A) = \alpha\},$$

$$\mathrm{Min}(\alpha) := \inf\{I_A : A \in \mathcal{A}, \mu(A) = \alpha\}, \qquad \alpha \in J.$$

Before we describe the sets for which these extremal values are attained, we state three simple lemmas.

Let $A^c := J^2 \setminus A$ be the *complement* of A.

LEMMA 2 (Complement Lemma). *For $A \in \mathcal{A}$ with $\mu(A) = \alpha$ we have*

$$I_A = I_{A^c} + 2\alpha - 1$$

and therefore (for $\alpha \in J$)

$$\mathrm{Min}(\alpha) = \mathrm{Min}(1-\alpha) + 2\alpha - 1 \quad \textit{and} \quad \mathrm{Max}(\alpha) = \mathrm{Max}(1-\alpha) + 2\alpha - 1.$$

PROOF. We have $H_{A^c}(x) = 1 - H_A(x)$ and $V_{A^c}(x) = 1 - V_A(x)$ which implies

$$I_{A^c} = \int_0^1 (1 - H_A(x))(1 - V_A(x))d\lambda(x)$$
$$= \int_0^1 \{1 - H_A(x) - V_A(x) + H_A(x)V_A(x)\}d\lambda(x)$$
$$= 1 - 2\alpha + I_A. \qquad \square$$

Note that the supremum (infimum) is attained in A for α iff the supremum (infimum) is attained in A^c for $1-\alpha$, so that we may assume $\alpha \leqq \frac{1}{2}$.

We call the sets $\{(x,x) \in J^2 : x \in J\}$, $\{(x, 1-x) \in J^2 : x \in J\}$ the *diagonal*, the *cross diagonal*, resp.

Let R_d, resp. R_c be reflection w.r.t. these diagonals. We call a transformation

$$(T \times T) : J^2 \to J^2$$

a *product isomorphism* if $T : J \to J$ is measurable, measure preserving and almost everywhere 1–1.

Lemma 3 (Reflection and Invariance Lemma). *For $A \in \mathcal{A}$ and for a product isomorphism $T \times T$ we have*

$$I_A = I_{R_dA} = I_{R_cA} = I_{(T\times T)A}.$$

Proof. We have $H_{R_dA} = V_A$, $H_{R_cA}(x) = V_A(1-x)$ and $H_{(T\times T)A}(x) = H_A(T^{-1}x)$ (and similar formulas for V_A) which imply the statement. $\square$

We will identify two sets A and B if $\mu(A \triangle B) = 0$, and we introduce the habitual metric d:

$$d(A,B) := \mu(A \triangle B), \qquad A, B \in \mathcal{A}.$$

Lemma 4 (Continuity Lemma). *For $A, B \in \mathcal{A}$ we have*

$$|I_A - I_B| \leqq 2\mu(A \triangle B)$$

and therefore (for $\alpha, \beta \in J$)

$$|\mathrm{Max}(\alpha) - \mathrm{Max}(\beta)| \leqq 2|\alpha - \beta| \quad \textit{and} \quad |\mathrm{Min}(\alpha) - \mathrm{Min}(\beta)| \leqq 2|\alpha - \beta|.$$

Proof. The first inequality follows from

$$|I_A - I_B| = \left| \int H_A(V_A - V_B) + V_B(H_A - H_B)d\lambda \right|$$
$$\leqq \int |V_A - V_B| d\lambda + \int |H_A - H_B| d\lambda$$
$$\leqq 2\mu(A \triangle B).$$

The second inequality follows by choosing for $\alpha > \beta$ a set A with measure α such that I_A is close to $\mathrm{Max}(\alpha)$, and a subset B of A with measure β. Then $\mu(A \triangle B) = \alpha - \beta$, and application of the first inequality yields the second inequality.

The third inequality follows analogously. □

3. The sets where the maximal and minimal 2-correlations are attained

We define the following sets for $0 \leqq \alpha \leqq \frac{1}{2}$:

$$A_\alpha^{\max} := ([0, 1 - \sqrt{1-\alpha}] \times [0, 1]) \cup ([1 - \sqrt{1-\alpha}, 1] \times [0, 1 - \sqrt{1-\alpha}]).$$

For $\alpha < \frac{1}{2}$, let

$$s := \frac{1 + \sqrt{1 - 2\alpha\left(\frac{N+1}{N}\right)}}{N+1}$$

where

$$N := \mathrm{int}\left(\frac{1}{1-2\alpha}\right)$$

is such that

$$\frac{1}{2} - \frac{1}{2N} \leqq \alpha < \frac{1}{2} - \frac{1}{2(N+1)}.$$

Now let

$$A_\alpha^{\min} := \{(x, y) \in J^2 : y \leqq s \cdot \mathrm{int}(x/s)\}$$

or equivalently

$$A_\alpha^{\min} := \bigcup_{i=1}^{N-1} ([is, (i+1)s] \times [0, is]) \cup ([Ns, 1] \times [0, Ns]).$$

Finally we define

$$A_{1/2}^{\min} := \{(x, y) \in J^2 : y \leqq x\}.$$

We call $A_\alpha^{\min}$ a *staircase set*.

Straightforward computations show that both $A_\alpha^{\max}$ and $A_\alpha^{\min}$ have measure α (see Figs. 1–11).

We call a set A with $\mu(A) = \alpha < \frac{1}{2}$ a *disturbed staircase set* if A is not a staircase set and if there exists a set B such that (N and s as above)

$$A = B \cup \bigcup_{i=1}^{N-1} ([x_i, x_{i+1}] \times [0, x_i])$$

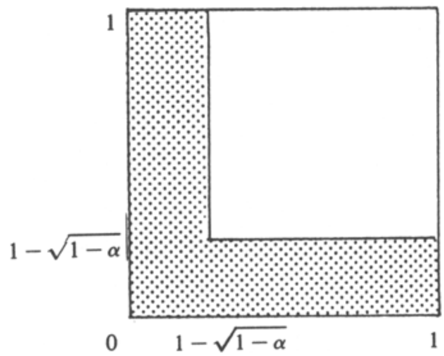

Fig. 1. $A_\alpha^{\max}$ $(0 \leqq \alpha \leqq \frac{1}{2})$.

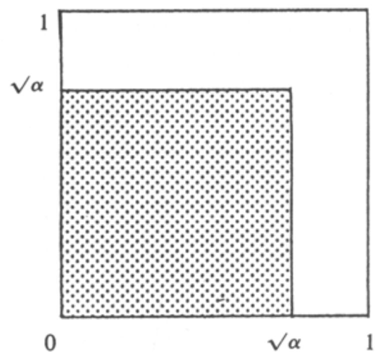

Fig. 2. $R_c(A_{1-\alpha}^{\max})^c$ $(\frac{1}{2} \leqq \alpha \leqq 1)$.

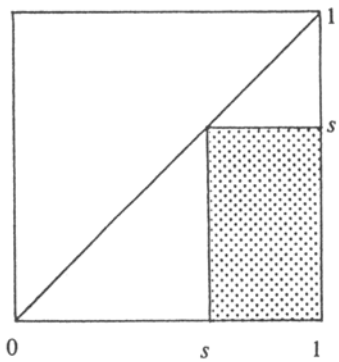

Fig. 3. $A_\alpha^{\min}$ $(0 \leqq \alpha < \frac{1}{4})$.

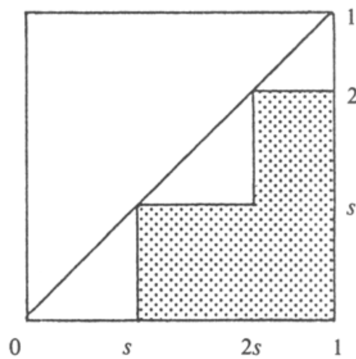

Fig. 4. $A_\alpha^{\min}$ $(\frac{1}{4} \leqq \alpha < \frac{1}{3})$.

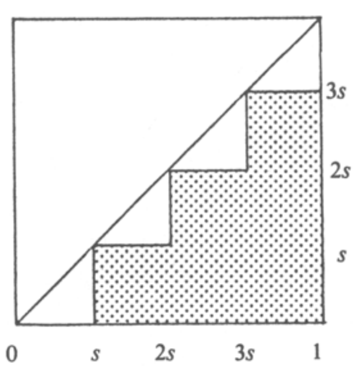

Fig. 5. $A_\alpha^{\min}$ $(\frac{1}{3} \leqq \alpha < \frac{3}{8})$.

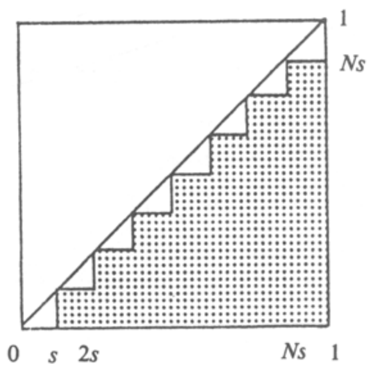

Fig. 6. $A_\alpha^{\min}$ $(\frac{1}{2} - 1/(2N) \leqq \alpha < \frac{1}{2} - 1/(2(N+1)))$.

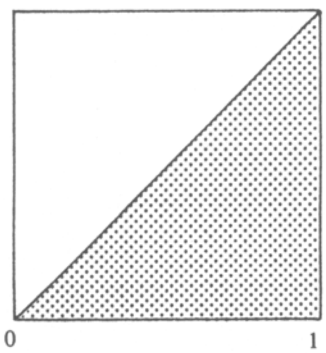

Fig. 7. $A^{\min}_{1/2}$.

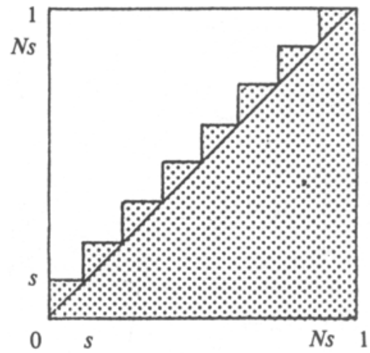

Fig. 8. $R_d(A^{\min}_{1-\alpha})^c$ $(\frac{1}{2} + 1/(2(N+1)) < \alpha \leqq \frac{1}{2} + 1/2N)$.

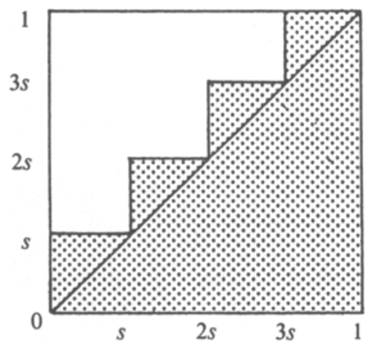

Fig. 9. $R_d(A^{\min}_{1-\alpha})^c$ $(\frac{5}{8} < \alpha \leqq \frac{2}{3})$.

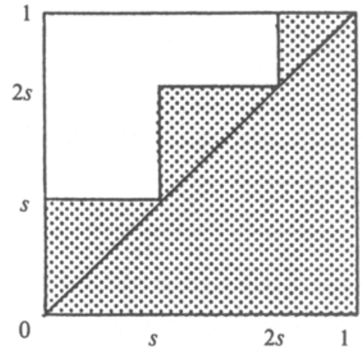

Fig. 10. $R_d(A^{\min}_{1-\alpha})^c$ $(\frac{2}{3} < \alpha \leqq \frac{3}{4})$.

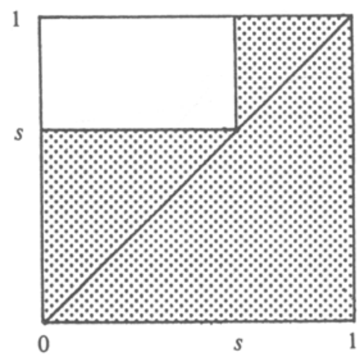

Fig. 11. $R_d(A^{\min}_{1-\alpha})^c$ $(\frac{3}{4} < \alpha \leqq 1)$.

with $0(=x_0) < x_1 < \cdots < x_{N-1} < x_N = 1$, and for some $i_0 \in \{0, 1, \ldots, N-1\}$

$$x_{i+1} - x_i = \begin{cases} 1-(N-1)s, & \text{if } i = i_0 \\ s, & \text{if } i \neq i_0, \quad i \in \{0, 1, \ldots, N-1\} \end{cases}$$

and, for some $0 < \gamma < x_{i_0+1} - x_{i_0}$, B is a subset of

$$[x_{i_0} + \gamma, x_{i_0+1}] \times [x_{i_0}, x_{i_0} + \gamma].$$

See Figs. 12–15.

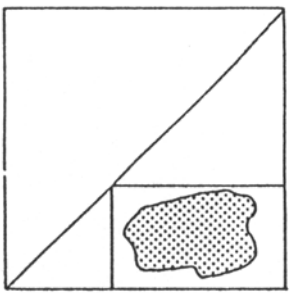

Fig. 12. Two disturbed staircase sets and the complements of the reflected (w.r.t. the diagonal) sets of two disturbed staircase sets.

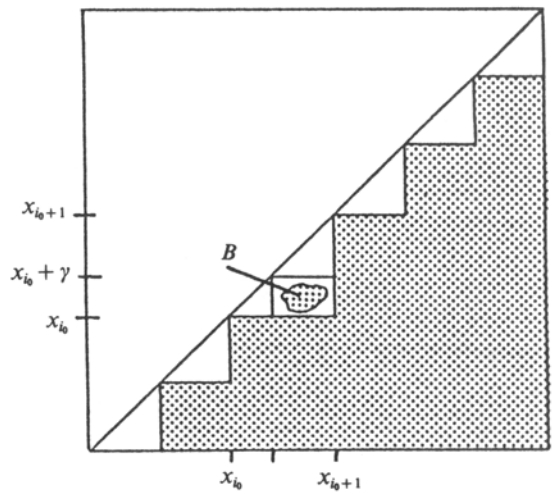

Fig. 13.

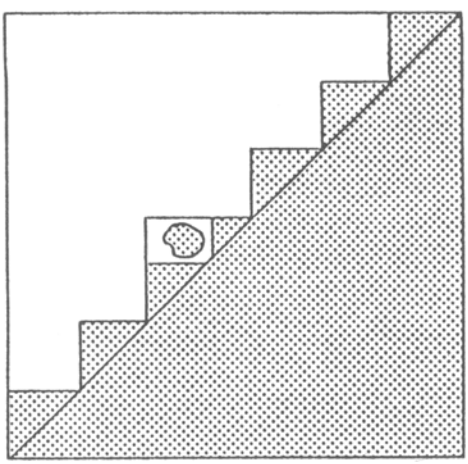

Fig. 14.

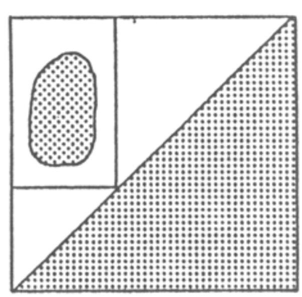

Fig. 15.

4. Results

THEOREM 1.

$$\mathrm{Max}(\alpha) = \begin{cases} 2\alpha - 1 + (1-\alpha)^{3/2}, & 0 \leqq \alpha \leqq \frac{1}{2} \\ \alpha^{3/2}, & \frac{1}{2} \leqq \alpha \leqq 1. \end{cases}$$

PROPOSITION 1. *This supremum is attained in the sets* $A_\alpha^{\max}$ *for* $0 \leqq \alpha \leqq \frac{1}{2}$ *and in* $(A_{1-\alpha}^{\max})^c$ *for* $\frac{1}{2} \leqq \alpha \leqq 1$.

Conversely, each set A with measure α *and* $I_A = \mathrm{Max}(\alpha)$ *is product isomorphic to one of the above-mentioned sets.*

For the proof of Theorem 1 we refer to Katz [Ka] or Finke [F] or Gandolfi, Keane and de Valk [GKV].

Proposition 1 is proved in [GKV].

THEOREM 2.

$$\mathrm{Min}(\alpha) = \begin{cases} \dfrac{(N-1)N}{6(N+1)^2}(1-2\delta)(1+\delta)^2, & \text{if } 0 \leqq \alpha < \frac{1}{2}, \\ \frac{1}{6}, & \text{if } \alpha = \frac{1}{2}, \\ 2\alpha - 1 + \mathrm{Min}(1-\alpha), & \text{if } \frac{1}{2} < \alpha \leqq 1, \end{cases}$$

with

$$N = \mathrm{int}\left(\frac{1}{1-2\alpha}\right) \quad \text{and} \quad \delta = \sqrt{1 - 2\alpha\left(\frac{N+1}{N}\right)}.$$

REMARK. For $\frac{1}{2} - 1/(2N) \leqq \alpha < \frac{1}{2} - 1/(2(N+1))$ we have $1/N \geqq \delta > 0$, so $\delta \to 0$ if $\alpha \to \frac{1}{2}$. Note that if $1/(1-2\alpha)$ is an integer we have

$$\mathrm{Min}(\alpha) = \mathrm{Min}\left(\frac{1}{2} \pm \frac{1}{2N}\right) = \frac{(N \pm 1)(N \pm 2)}{6N^2} = \frac{\alpha(4\alpha - 1)}{3}$$

and in these points the function Min has a left derivative which is smaller than the right derivative. For the function Max this phenomenon only occurs at $\alpha = \frac{1}{2}$. (See Fig. 16.)

Note further that $\mathrm{Min}(\alpha) \geqq \alpha(4\alpha - 1)/3$ for all $\alpha \in J$.

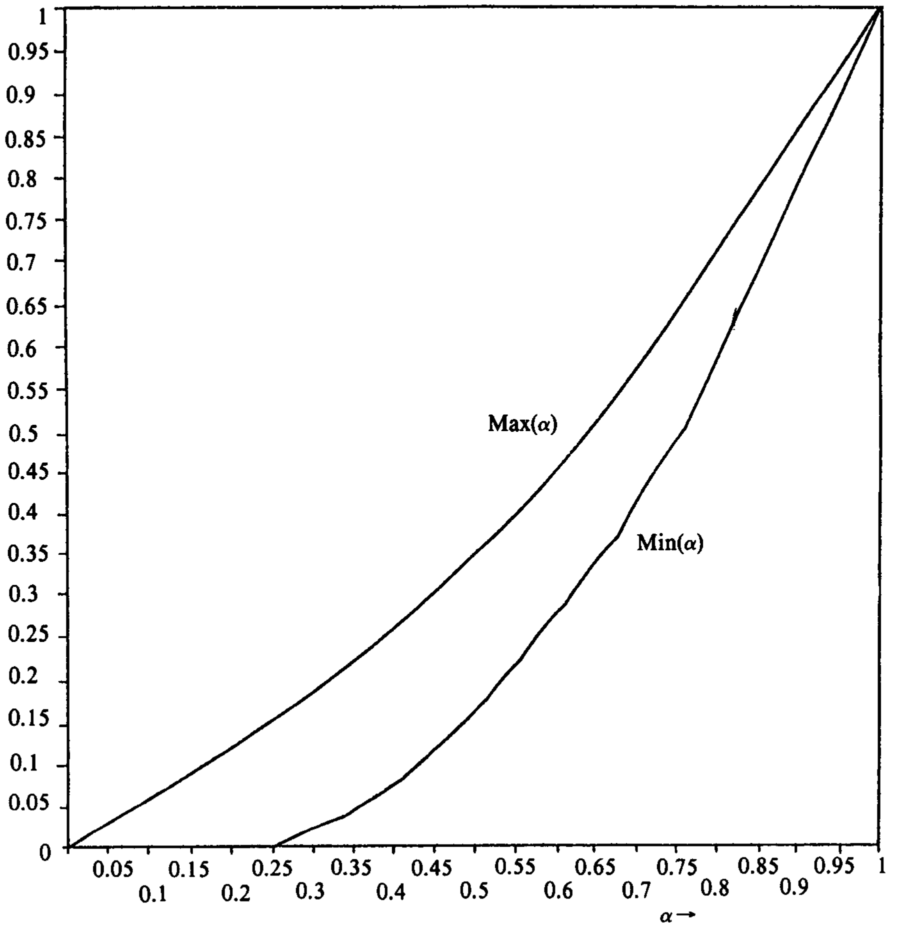

Fig. 16. The functions Max and Min.

Proposition 2. *The infimum is attained in the staircase sets* $A_\alpha^{\min}$ *for* $0 \leqq \alpha \leqq \frac{1}{2}$, $(A_{1-\alpha}^{\min})^c$ *for* $\frac{1}{2} \leqq \alpha \leqq 1$, *and it is also attained in the disturbed staircase sets for* $\alpha < \frac{1}{2}$ *and in the complements of these for* $\alpha > \frac{1}{2}$.

Conversely, when $1/(1-2\alpha)$ *is an integer or* $\alpha = \frac{1}{2}$, *if the infimum is attained in some set* $A \in \mathcal{A}$ *with measure* α, *then* A *is product isomorphic to a staircase set* ($\alpha \leqq \frac{1}{2}$), *or to the complement of a staircase set* ($\alpha > \frac{1}{2}$).

When $\alpha \neq \frac{1}{2}$ *and* $1/(1-2\alpha)$ *is not an integer, if the infimum is attained in some set* $A \in \mathcal{A}$ *with measure* α, *then* A *is product isomorphic to a staircase set or to a disturbed staircase set* ($\alpha < \frac{1}{2}$) *or to the complement of one of these sets* ($\alpha > \frac{1}{2}$).

We prove Theorem 2 in Section 5 and we prove Proposition 2 in Section 6.

5. Proof of Theorem 2

Let $\alpha > 0$ be fixed. In six steps we will, by various rearrangement procedures, gradually diminish the size of the collection of sets A for which $I_A = \mathrm{Min}(\alpha)$, until we reach the staircase sets, so proving the statement of Theorem 2.

Step 1. Standardization

By the continuity lemma we may approximate a set A ($\mu(A) = \alpha$) by a finite union of squares of the form $[x, x + \delta) \times [y, y + \delta)$ with $x, y \in J$, where $\delta > 0$ is the reciprocal of an integer.

Then H_A and V_A are constant on intervals. We rearrange J with a transformation T (a permutation of intervals) such that $H_{(T \times T)A}$ is non-increasing (see Figs. 17 and 18). We use the notation $\tau := T \times T$.

The Invariance Lemma implies that $I_{\tau A} = I_A$.

We say that a set is in *standard form* if it is the set under (the graph of) a non-decreasing function. We will obtain from τA a set A' in standard form with $I_{A'} \leqq I_{\tau A}$. This is accomplished by moving squares horizontally to the right.

If τA is not in standard form, then there exist squares S_1 and S_2 such that

$$S_1 := [x_1, x_1 + \delta) \times [y, y + \delta) \text{ is a subset of } \tau A,$$

$$S_2 := [x_2, x_2 + \delta) \times [y, y + \delta) \text{ is disjoint with } \tau A,$$

for some $x_1 < x_2$. Define the set $\sigma\tau A$ (Fig. 19):

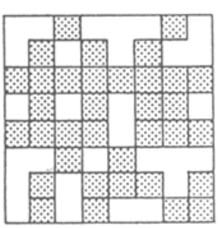

Fig. 17. A.

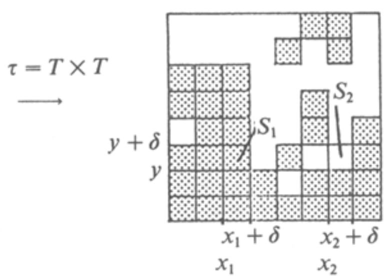

Fig. 18. τA.

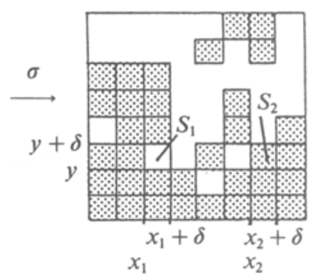

Fig. 19. $\sigma\tau A$.

$$\sigma\tau A := (\tau A \setminus S_1) \cup S_2;$$

then $\mu(\sigma\tau A) = \mu(\tau A)$ and we will prove that $I_{\sigma\tau A} \leqq I_{\tau A}$.

We have $H_{\sigma\tau A} = H_{\tau A} =: H$ and

$$V_{\sigma\tau A}(x)=\begin{cases}V_{\tau A}(x)-\delta, & \text{if } x\in[x_1,x_1+\delta),\\ V_{\tau A}(x)+\delta, & \text{if } x\in[x_2,x_2+\delta),\\ V_{\tau A}(x), & \text{else.}\end{cases}$$

Therefore

$$\begin{aligned}I_{\tau A}-I_{\sigma\tau A}&=\int_{x_1}^{x_1+\delta}\delta H(x)dx-\int_{x_2}^{x_2+\delta}\delta H(x)dx\\ &=\delta^2\{H(x_1)-H(x_2)\}\\ &\geqq 0.\end{aligned}$$

Note that we have equality iff H is constant on $[x_1,x_2+\delta)$. The set A' (in standard form) is obtained from A by applying τ (once) and a finite number of shifts of the type σ. Using these facts we obtain the next claim, in which we introduce the notation f_A and A_f (to stress the correspondence between a non-decreasing function f and a set A in standard form that is the set under f).

Claim 1 (Standardization).

$$\mathrm{Min}(\alpha)=\inf\{I_A:\mu(A)=\alpha,\ A=A_f\in\mathcal{A} \text{ in standard form, } f_A \text{ finite valued}\}\qquad(\alpha\in J).$$

Remark. From Helly's selection principle ([Luk] par. 3.5) it follows that the infimum is actually attained in some set in standard form.

Step 2. (*Under the diagonal*)

Because of the Complement Lemma we assume that $\alpha<\frac{1}{2}$. It is easy to see that $\mathrm{Min}(\alpha)=0$ for $\alpha\leqq\frac{1}{4}$; take e.g. $A=[1-\sqrt{\alpha},1]\times[0,\sqrt{\alpha}]$.

Therefore we assume further in this proof that $\frac{1}{4}<\alpha<\frac{1}{2}$.

Take a set A in standard form with measure α and such that f_A is finite valued. Assume that A does not lie under the diagonal (a set A lies *under the diagonal* if A is a subset of $A_{1/2}^{\min}$). We will transform A to a set lying under the diagonal such that I_A does not increase.

Let A be a union of $\delta\times\delta$ squares. We choose

$$S_1:=[x_1,x_1+\delta)\times[y_1,y_1+\delta)\text{ subset of } A$$

and

$$S_2:=[x_2,x_2+\delta)\times[y_2,y_2+\delta)\text{ disjoint with } A$$

such that S_1 lies above the diagonal and S_2 lies under the diagonal (by passing from δ to $\frac{1}{2}\delta$ we may assume that there exist such squares entirely above or under the diagonal), and such that

$$f_A(x_1-)\leqq y_1, \qquad f_A(x_2+\delta+)\geqq y_2+\delta$$

(these conditions guarantee that the transformed set will be in standard form). Let g be such that

$$A_g=(A_f\setminus S_1)\cup S_2.$$

We will prove that $I_{A_g}<I_{A_f}$.

We say that a rectangle $[x',x'')\times[y',y'')$ (disjoint with the diagonal and a subset of a set A in standard form) *interferes* with the horizontal sections $H_A(x)$ with $x'\leqq x<x''$ and with the vertical sections $V_A(y)$ with $y'\leqq y<y''$.

We introduce this definition because the removal of this rectangle from A decreases I_A by the amount (as follows from the computation in this step)

$$(y''-y')\cdot(x''-x')\cdot\left\{\frac{\int_{x'}^{x''}H_A(x)dx}{(x''-x')}+\frac{\int_{y'}^{y''}V_A(y)dy}{(y''-y')}\right\},$$

i.e., the change in I_A equals the area of the rectangle times the average value of the sections with which the rectangle interferes.

The intuitive idea behind the inequality $I_{A_g}<I_{A_f}$ is the fact that the square S_1 interferes with the sections marked with a $-$ sign and the square S_2 interferes with the sections marked with a $+$ sign (in Fig. 20); the first total is larger than the second. We have

$$\begin{aligned}I_{A_f}-I_{A_f\setminus S_1}&=\int_{x_1}^{x_1+\delta}H_A(x)\delta dx+\int_{y_1}^{y_1+\delta}V_A(y)\delta dy\\&\geqq\delta\cdot(1-x_1)\cdot\delta+\delta\cdot(y_1+\delta)\cdot\delta\end{aligned}$$

and analogously

$$\begin{aligned}I_{A_f\setminus S_1}-I_{A_g}&=-\int_{x_2}^{x_2+\delta}H_A(x)\delta dx-\int_{y_2}^{y_2+\delta}V_A(y)\delta dy\\&\geqq-\delta\cdot(1-x_2-\delta)\cdot\delta-\delta y_2\delta\end{aligned}$$

which implies

$$I_{A_f}-I_{A_g}\geqq\delta^2\{y_1-x_1+x_2-y_2+2\delta\}\geqq 4\delta^3>0$$

since $x_1+\delta\leqq y_1$ and $x_2\geqq y_2+\delta$.

We write our conclusions in the next claim:

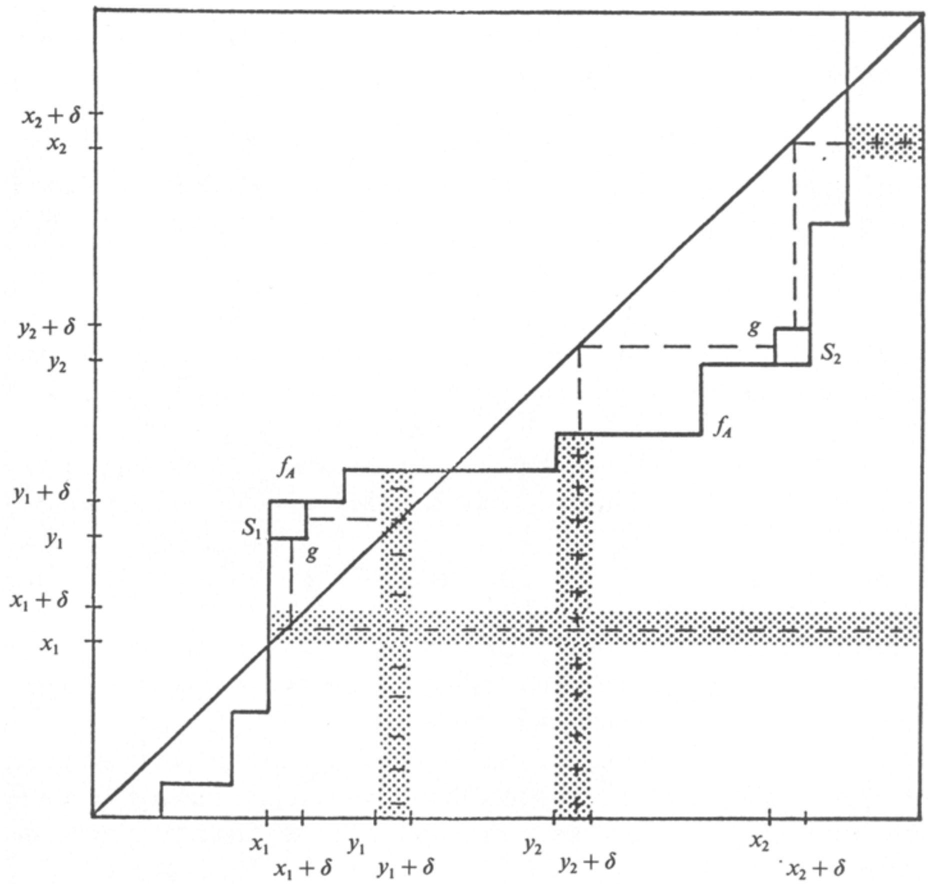

Fig. 20.

Claim 2 (Under the diagonal). For $\frac{1}{4} < \alpha < \frac{1}{2}$ we have

$$\operatorname{Min}(\alpha) = \inf\{I_A : A \in \mathcal{A}, \mu(A) = \alpha, A \text{ in standard form}, \\ A \text{ under the diagonal}, f_A \text{ finite valued}\}.$$

Lemma 5 (Windowing). *Let $f_A : J \to J$ be a non-decreasing function such that $f_A(a) = a$, $f_A(b) = b$ for some $0 \leqq a < b \leqq 1$. Define*

$$A^w := A \cap ([a, b] \times [a, b])$$

and let H^w and V^w be the corresponding sections on $[a, b]$ (Fig. 21),

$$H^w := H_A - (1 - b) \quad \text{and} \quad V^w := V_A - a.$$

Let $\alpha^w := \mu(A^w)$ and

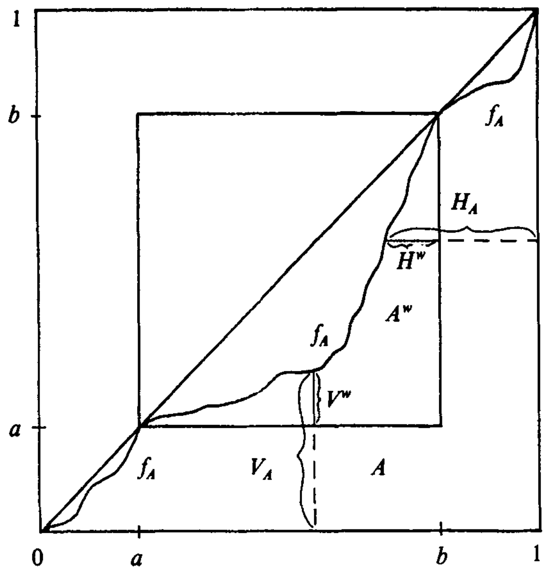

Fig. 21.

$$I_{A^w} = \int_a^b H^w(x)V^w(x)dx,$$

then

$$I_A = I_{A^w} + \int_{[0,a]\cup[b,1]} H_A(x)V_A(x)dx + (1-b+a)\alpha^w + (b-a)(1-b)a.$$

PROOF. We have

$$\begin{aligned} I_A - \int_{[0,a]\cup[b,1]} H_A(x)V_A(x)dx &= \int_a^b (H^w(x)+1-b)(V^w(x)+a)dx \\ &= \int_a^b H^w(x)V^w(x)dx + (1-b)\int_a^b V^w(x)dx \\ &\quad + a\int_a^b H^w(x)dx + (b-a)(1-b)a \\ &= I_{A^w} + (1-b+a)\alpha^w + (b-a)(1-b)a. \quad \square \end{aligned}$$

COROLLARY. *Assume f_A is as in this lemma, and we change f_A on (a,b)* area preservingly *to f_B (i.e. $\mu(A)=\mu(B)$). Then I_A will change by the same amount as I_{A^w}.*

Step 3. (*Moving to the diagonal*)

Let A be a set in standard form, lying under the diagonal, such that for some positive integer N

$$f_A = \sum_{i=1}^{N} y_i \cdot 1_{[x_i, x_{i+1})}$$

with $0 = x_0 < x_1 < \cdots < x_{N+1} = 1$.

Let $d_i := x_i - x_{i-1}$ and $c_i := y_i - y_{i-1}$ $(i = 1, \ldots, N+1)$. Assume that

$$U(f_A) := \operatorname{card}\{i : y_i < x_i\} > 0,$$

then we will prove the existence of a set B in standard form, lying under the diagonal, with the same measure as A, and with a finite valued function f_B such that $I_B \leqq I_A$ and $U(f_B) < U(f_A)$.

We first give an intuitive sketch of our procedure (cf. Figs. 22 and 23). Let i be the first index such that $y_i < x_i$. We will change f_A on $[x_{i-1}, x_{i+1})$. Because of the Windowing Lemma we may restrict our attention to the square $[x_{i-1}, 1] \times [x_{i-1}, 1]$.

We transform the rectangle $[x_i, x_{i+1}) \times [y_{i-1}, y_i)$ (with area $d_{i+1} \cdot c_i$) such that $U(f_A)$ reduces by one. We change it to a rectangle with height $c_i + c_{i+1}$. This is possible if (Case I, Figs. 22 and 23)

$$d_{i+1} \cdot c_i \leqq (x_{i+1} - y_{i+1}) \cdot (c_i + c_{i+1}).$$

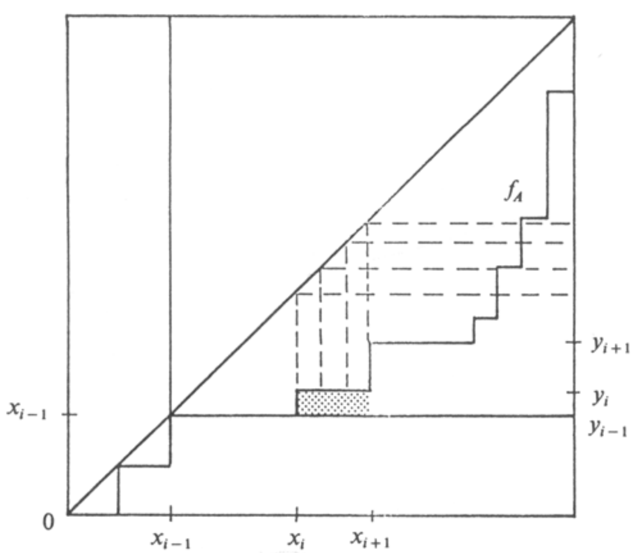

Fig. 22. Case I. Before the transformation.

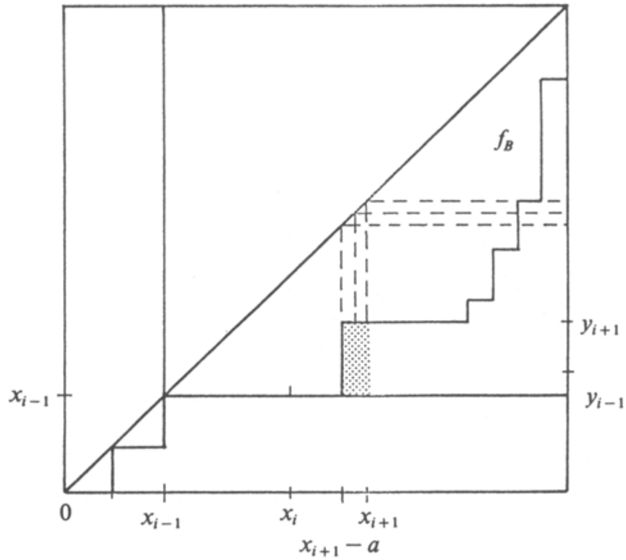

Fig. 23. Case I. After the transformation.

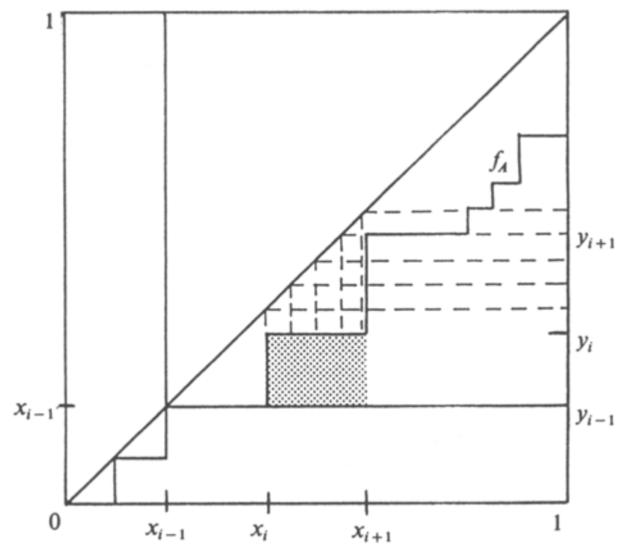

Fig. 24. Case II. Before the transformation.

Otherwise (Case II, Figs. 24 and 25) we transform it to a rectangle that has one corner on the diagonal and lies as far as possible to the right. The rectangle with area $d_{i+1} \cdot c_i$ interferes before the transformation with the set of horizontal sections $H_A(x)$, $x_i \leqq x < x_{i+1}$ and after the transformation it interferes with

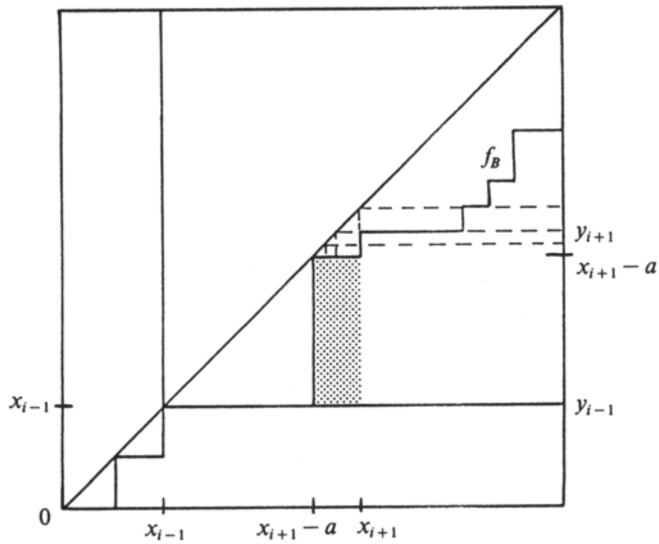

Fig. 25. Case II. After the transformation.

the subset of horizontal sections $H_A(x)$, $x_{i+1}-a \leqq x < x_{i+1}$ (in Case I, $a := d_{i+1}\cdot c_i/(c_i+c_{i+1})$; in Case II, choose $0<a<d_{i+1}$ such that $a\cdot(d_i+d_{i+1}-a)=d_{i+1}\cdot c_i$).

Because the first set contains some large sections, which are not contained in the subset, this subset has a smaller average value (see definition of interference). This crucial observation implies that I_A will decrease. (Note that the vertical sections V^w with which the described rectangles interfere, have length zero.)

Case I. $(d_{i+1}\cdot c_i \leqq (x_{i+1}-y_{i+1})\cdot(c_i+c_{i+1}))$

Replacing $[x_i, x_{i+1})\times[y_{i-1}, y_i)$ by $[x_{i+1}-a, x_{i+1})\times[y_{i-1}, y_{i+1})$ we have

$$\begin{aligned} I_{A_f}-I_B &= \int_{x_i}^{x_{i+1}} H^w(x)c_i dx - \int_{x_{i+1}-a}^{x_{i+1}} H^w(x)(c_i+c_{i+1})dx \\ &= \int_{x_i}^{x_{i+1}-a} H^w(x)c_i dx - \int_{x_{i+1}-a}^{x_{i+1}} H^w(x)c_{i+1}dx \\ &\geqq (d_{i+1}-a)\cdot H^w(x_{i+1}-a)\cdot c_i - a\cdot H^w(x_{i+1}-a)\cdot c_{i+1} \\ &= 0. \end{aligned}$$

Case II. $(d_{i+1}c_i \geqq (x_{i+1}-y_{i+1})\cdot(c_i+c_{i+1}))$

Replacing $[x_i, x_{i+1})\times[y_{i-1}, y_i)$ by $[x_{i+1}-a, x_{i+1})\times[y_{i-1}, x_{i+1}-a)$ we have

$$
\begin{aligned}
I_{A_f} - I_B &= \int_{x_i}^{x_{i+1}} H^w(x)c_i dx - \int_{x_{i+1}-a}^{x_{i+1}} H^w(x)(x_{i+1} - a - y_{i-1})dx \\
&= \int_{x_i}^{x_{i+1}-a} H^w(x)c_i dx - \int_{x_{i+1}-a}^{x_{i+1}} H^w(x)(x_{i+1} - a - y_{i-1} - c_i)dx \\
&\geqq (d_{i+1} - a)\cdot H^w(x_{i+1} - a)\cdot c_i \\
&\quad - a \cdot H^w(x_{i+1} - a)\cdot(x_{i+1} - a - y_{i-1} - c_i) \\
&= 0.
\end{aligned}
$$

It is easy to see that we can reduce $U(f_A)$ to zero, while I_A does not increase, and we conclude

Claim 3 (Moving to the diagonal). For $\frac{1}{4} < \alpha < \frac{1}{2}$ we have

$$
\mathrm{Min}(\alpha) = \inf\left\{ I_A : A \in \mathcal{A}, \mu(A) = \alpha, \text{ for some } N \in \mathbf{N}, f_A = \sum_{i=1}^{N} x_i \cdot 1_{[x_i, x_{i+1})}, \right.
$$
$$
\left. 0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1 \right\}.
$$

Step 4. (*Rearrangement*)

We will prove that we may assume that $(d_i)_{i=1}^{N+1}$ is a non-increasing sequence. Let A be a set as in Claim 3 and assume that for some $i \in \{1, \ldots, N\}$ we have

$$d_i < d_{i+1}.$$

We will change f_A on $[x_i, x_{i+1})$ area-preservingly such that for the new function g we have $d_i' > d_{i+1}'$ and $I_{A_f} = I_{A_g}$.

The intuitive idea behind this equality is the fact that both rectangles with area $d_{i+1} \cdot d_i$ interfere with horizontal sections of the same constant length (see Fig. 26). Because of the Windowing Lemma we may restrict our attention to

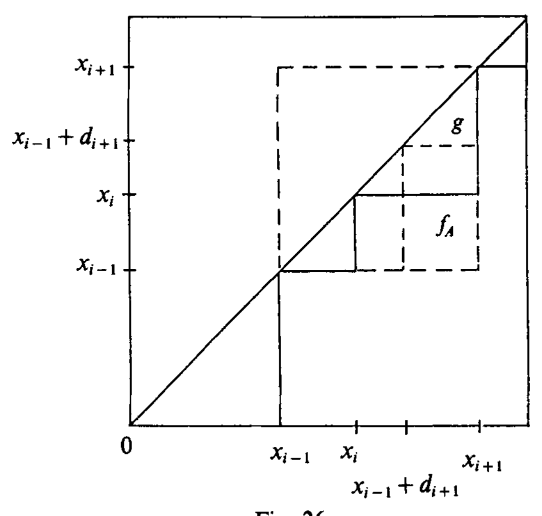

Fig. 26.

$[x_{i-1}, x_{i+1}) \times [x_{i-1}, x_{i+1})$. We replace $[x_{i-1}+d_i, x_{i+1}) \times [x_{i-1}, x_{i-1}+d_i)$ by $[x_{i-1}+d_{i+1}, x_{i+1}) \times [x_{i-1}, x_{i-1}+d_{i+1})$ and it is trivial that

$$I_{A_f} = I_{A_g} \qquad \text{(because } H^w = 0 \text{ or } V^w = 0\text{)}.$$

We conclude

CLAIM 4 (Rearrangement). For $\frac{1}{4} < \alpha < \frac{1}{2}$ we have

$$\mathrm{Min}(\alpha) = \inf \Big\{ I_A : A \in \mathcal{A}, \mu(A) = \alpha, \text{for some } N \in \mathbf{N}, f_A = \sum_{i=1}^{N} x_i \cdot 1_{[x_i, x_{i+1})},$$
$$0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1,$$
$$d_1 \geqq d_2 \geqq \cdots \geqq d_N \geqq d_{N+1} \Big\}.$$

Step 5. (*Equality of Differences*)

Let A be as in Claim 4. We will prove that we may assume that

$$d_1 = d_2 = \cdots = d_N \geqq d_{N+1}.$$

Assume that for some $i \in \{1, \ldots, N-1\}$ we have

$$d_i > d_{i+1} \geqq d_{i+2}.$$

We will change f_A area-preservingly to g on $[x_{i-1}, x_{i+2})$ (cf. Fig. 27). Because

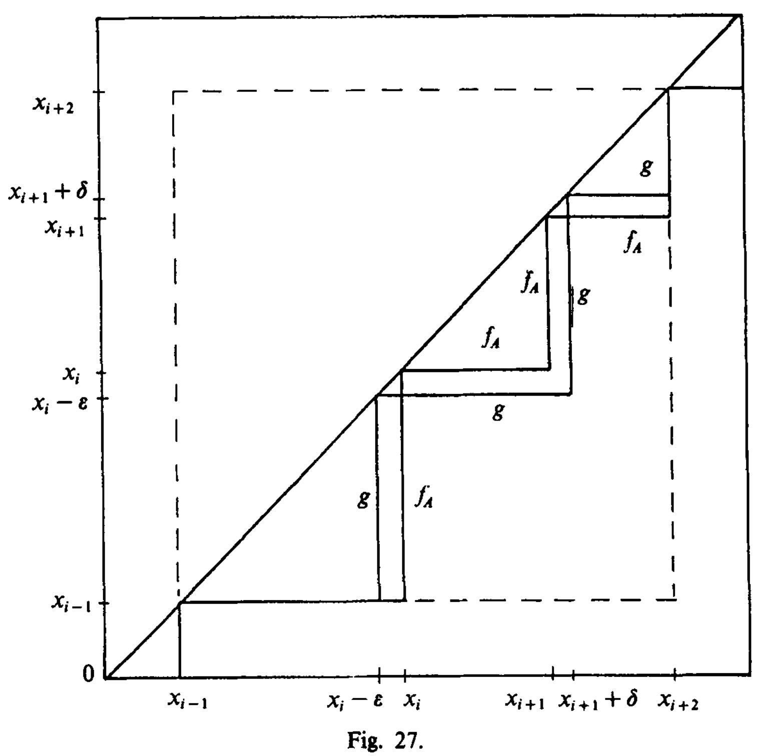

Fig. 27.

of the Windowing Lemma we may restrict our attention to $[x_{i-1}, x_{i+2}) \times [x_{i-1}, x_{i+2})$. We will obtain $I_{A_g} < I_{A_f}$.

Let $0 < \varepsilon < d_i - d_{i+1}$. Since $d_i > d_{i+1} \geqq d_{i+2}$ we can find $\delta > 0$ such that

$$(*) \qquad \varepsilon(d_i - \varepsilon) + \delta(d_{i+2} - \delta) = \varepsilon d_{i+1} + \delta d_{i+1} + \varepsilon\delta.$$

Define

$$d_i' := d_i - \varepsilon, \quad d_{i+1}' := d_{i+1} + \varepsilon + \delta, \quad d_{i+2}' := d_{i+2} - \delta,$$

and let g be the changed version of f (corresponding to d_i', d_{i+1}', d_{i+2}'), then

$$\begin{aligned} I_{A_f} - I_{A_g} &= d_{i+1} \cdot d_i \cdot d_{i+2} - d_{i+1}' \cdot d_i' \cdot d_{i+2}' \qquad (\text{use } (*)) \\ &= \varepsilon(d_i - d_{i+2})(d_i - d_{i+1}) - \varepsilon^2(d_i - d_{i+2}) - \varepsilon\delta(d_{i+1} - d_{i+2} + \varepsilon + \delta) \end{aligned}$$

and this is positive if ε is small enough ($\delta \to 0$ if $\varepsilon \to 0$). We conclude

Claim 5 (Equality of Differences). For $\frac{1}{4} < \alpha < \frac{1}{2}$ we have

$$\begin{aligned} \operatorname{Min}(\alpha) = \inf \Big\{ I_A : A \in \mathcal{A}, \mu(A) = \alpha, \text{ for some } N \in \mathbf{N}, f_A = \sum_{i=1}^{N} x_i \cdot 1_{[x_i, x_{i+1})}, \\ 0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1, \\ d_1 = \cdots = d_N \geqq d_{N+1} \Big\}. \end{aligned}$$

Step 6. (*Conclusion*)

These computations will prove Theorem 2. Let A be as in Claim 5, and set $s := d_1 = \cdots = d_N$. We have $1/(N+1) \leqq s \leqq 1/N$ and (see e.g. Fig. 6)

$$\alpha = \mu(A) = \sum_{i=1}^{N-1} s \cdot is + (1 - Ns) \cdot Ns = Ns - \frac{N(N+1)s^2}{2},$$

this implies

$$s = \frac{1 + \sqrt{1 - 2\alpha\left(\frac{N+1}{N}\right)}}{N+1} \qquad \left(+ \text{ sign because } s \geqq \frac{1}{N+1} \right).$$

Further, for I_A we have

$$I_A = \sum_{i=1}^{N-1} s \cdot is(1 - is - s) = \frac{s^2 \cdot N(N-1)}{6} \cdot \{3 - 2(N+1)s\}$$

$$= \frac{(N-1)N}{6(N+1)^2} \cdot (1 - 2\delta)(1+\delta)^2,$$

when we write

$$\delta := \sqrt{1 - 2\alpha\left(\frac{N+1}{N}\right)}.$$

This is the formula for $\mathrm{Min}(\alpha)$ in Theorem 2. We have computed $\mathrm{Min}(\alpha)$ for $\alpha < \frac{1}{2}$. The continuity of $\mathrm{Min}(\alpha)$ leads to $\mathrm{Min}(\frac{1}{2}) = \frac{1}{6}$ (use $N \to \infty$ and $\delta \to 0$ if $\alpha \to \frac{1}{2}$).

The Complement Lemma leads to the formula for $\alpha > \frac{1}{2}$. □

6. Proof of Proposition 2

In Step 6 of Section 5 we proved that the infimum is attained in the staircase sets. A straightforward computation shows that the infimum is also attained in the disturbed staircase sets. Observe that the subset B (in the definition of a disturbed staircase set in Section 3) interferes with sections of the same size as a rectangle.

Let $A \in \mathcal{A}$ be a set with measure $\alpha < \frac{1}{2}$, where the infimum is attained. We can generalize Steps 1–5 of the proof of Theorem 2 to A with $I_A = \mathrm{Min}(\alpha)$. Two integrable functions $f, g : [0, \infty) \to [0, \infty)$ are called *equimeasurable* (see [HLP] par. 10.12) if

$$\lambda\{x : f(x) \geqq y\} = \lambda\{x : g(x) \geqq y\} \qquad \text{for all } y > 0.$$

Let f: $[0, \infty) \to [0, \infty)$ be an integrable function. It is a well-known fact that there exists a non-increasing function g (the so-called *equimeasurable decreasing rearrangement* of f) such that f and g are equimeasurable.

Let H_A^* be the equimeasurable decreasing rearrangement of H_A. We define

$$A_1 := \{(x, y) \in J^2 : 1 - H_A^*(y) \leqq x\}.$$

Then A_1 is a set in standard form and this method of standardization generalizes Step 1. A simple approximation argument yields $I_{A_1} \leqq I_A$, but $I_{A_1} = I_A$ because the infimum is attained in A.

If A_1 does not lie under the diagonal, then we can strictly reduce I_{A_1} (and obtain a set A_2) by moving a part of A_1 lying above the diagonal to a place under the diagonal, as a slight modification of Step 2 shows (consider the interference in Fig. 20). Therefore we may assume $A_1 = A_2$, and this set lies under the diagonal.

A modification of Step 3 (approximation by stepfunctions) transforms A_2 to a set of the type in Claim 3, such that $I_{A_3} \leqq I_{A_2}$ (but again $I_{A_3} = I_{A_2}$).

Application of Steps 4 and 5 (unmodified) leads to sets A_4 and A_5 of the type in Claim 4, Claim 5, resp. with $I_{A_3} = I_{A_4} = I_{A_5}$.

We consider A_5 and go backwards to determine what A can be. If A_4 is not of the type in Claim 5, then the computation in Step 5 would imply that $I_{A_5} < I_{A_4}$. So $A_4 = A_5$.

Because rearrangement does not change I_A, we conclude that A_3 is of the type in Claim 3 with $d_i = d$ for all $i \neq i_0$ (for some i_0). Because $I_{A_2} = I_{A_3}$, the set A_2 ($= A_1$) is a staircase set or a disturbed staircase set in standard form (see the interference in Figs. 24 and 25). Note that the edgepoints (x_i, x_i) $(i \neq i_0 - 1)$ cannot be removed from the diagonal without changing the measure of the set. We consider the effect of moving some subset of A_1 horizontally to the left. If the new set is still a staircase set or a disturbed staircase set, then I_{A_1} will not change. But if the new set is no longer of this type, then I_{A_1} will change as a step-1-type computation shows (consider the interference with the horizontal sections). So before the process of moving rectangles to the right (as in Step 1), we already had a staircase set or a disturbed one. So A is product isomorphic to a set of this type. This proves Proposition 2 for the case $\alpha < \frac{1}{2}$.

The case $\alpha = \frac{1}{2}$ can be proved analogously. The case $\alpha > \frac{1}{2}$ follows from the case $\alpha < \frac{1}{2}$ (use the Complement Lemma). □

7. Remarks

(1) Katz proved a kind of symmetrization theorem ([Ka], Th. 3, p. 66) for the maximum case; for each set A, which is the set under (the graph of) a non-increasing function f_A (standard form in maximum case) and which is not symmetric (w.r.t. the diagonal), there exists a symmetric set A^{SYM} (in standard form with the same measure as A) such that

$$I_{A^{\mathrm{SYM}}} > I_A.$$

A^{SYM} is obtained in the following way. Let

$$x_0 := \sup\{x \in J : f_A(x) > x\},$$

let $f_{R_d(A)}$ be the function corresponding to $R_d(A)$, let

$$g := \tfrac{1}{2}(f_A + f_{R_d(A)}) \cdot 1_{[0,x_0]},$$

let C_g be the set under g, define A^{SYM} as

$$A^{\mathrm{SYM}} := C_g \cup R_d(C_g).$$

(See Fig. 28.) This symmetrization method does not work in the minimum case; i.e., given a set A (set under a non-decreasing function) we can construct a set A'^{SYM} (symmetric w.r.t. the cross diagonal) in an analogous way, but we will not always have $I_{A'^{\mathrm{SYM}}} \leqq I_A$, as the next counterexample shows.

Let $f = \frac{2}{5} \cdot 1_{[2/5,4/5]} + \frac{4}{5} \cdot 1_{[4/5,1)}$, then $\alpha = \frac{8}{25}$ and $I_{A_f} = 0.032 = \mathrm{Min}(\alpha)$, but $I_{A'^{\mathrm{SYM}}} = 0.036$. The infimum is attained in A, but not in A'^{SYM}, which does not touch the diagonal in each step (terminology from Step 3).

(2) Extension of the problem from J^2 to $\mathbf{R}^2$ is not possible. Given a set $A \subset \mathbf{R}^2$ with measure α we can define H_A and V_A in the usual way, but the problem is that

$$\int_{-\infty}^{+\infty} H_A(x) V_A(x) dx$$

can diverge. So the supremum is infinite. Further, the infimum is zero (take e.g. $A \subset (0, \infty) \times (-\infty, 0)$).

(3) In the minimum case there exists a continuous (w.r.t. d) mapping

$$F : J \to A$$

such that

$$I_{F(\alpha)} = \mathrm{Min}(\alpha) \qquad \text{for } \alpha \in J.$$

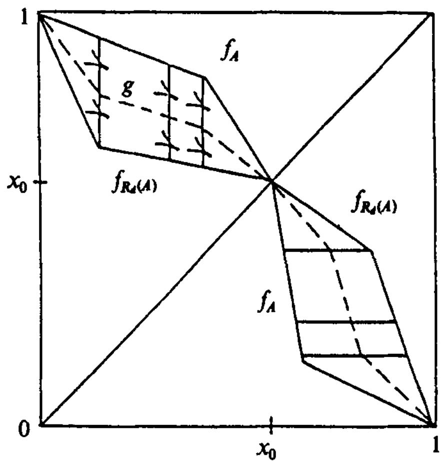

Fig. 28.

In the maximum case such a mapping is discontinuous in $\alpha = \frac{1}{2}$. For $\alpha < \frac{1}{2}$ we have (independent of the choice of the set A)

$$\mathrm{Range}(H_A) = \{1 - \sqrt{1-\alpha}, 1\}$$

and for $\alpha > \frac{1}{2}$

$$\mathrm{Range}(H_A) = \{\sqrt{\alpha}, 0\}.$$

If F could be chosen to be continuous, then the range of H_A would depend continuously on α.

In other words: in the maximum case the sections of $A_{1/2}^{\max}$ and $(A_{1/2}^{\max})^c$ have different ranges and in the minimum case the sections of $A_{1/2}^{\min}$ and $(A_{1/2}^{\min})^c$ have the same range.

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